Inverse analysis of an embankment using the Ensemble Kalman Filter including heterogeneity of the soft soil

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ABSTRACT: Geomechanical models are indispensable for reliable design of engineering structures and processes and hazard and risk evaluation. Model predictions are however far from perfect. Errors are introduced by fluctuations in the input or by poorly known parameters in the model. To overcome these problems an inverse modelling technique to incorporate measurements into the deterministic model to improve the model results can be implemented. This allows for observations of on-going processes to be used for enhancing the quality of subsequent model predictions.

In geomechanics several examples of inverse modelling exist where the improved model of the system is obtained by minimizing the discrepancy between the observed values in the system and the modelled state of the system within a time interval. This requires the implementation of the adjoint model. Even with the use of the adjoint compilers that have become available recently, this is a tremendous programming effort for the existing geomechanical model system.

The Ensemble Kalman filter is being implemented to overcome this problem. The Ensemble Kalman filter analyses the state of the system each time data becomes available. The Random Finite Element Method is used to simulate the heterogeneity of the subsurface.

Very promising results of a conceptual example, based on the construction of a road embankment on soft clay, are presented. The Ensemble Kalman filter is not only used for a straight forward identification of the elastic modulus E and the Ko parameter of several soil layers below the embankment, but also incorporates by means of the Random Finite Element Method the variation of the parameters throughout a soil layer.

1 INTRODUCTION

1.1 *Inverse modelling*

In geotechnical engineering context inverse modelling or back analysis consists in finding the values of the mechanical parameters, or of other quantities characterizing a soil or rock mass, that when introduced in the stress analysis of the problem under examination lead to results (e.g. displacements, stresses etc.) as close as possible to the corresponding in situ measurements. The optimal state of the system is obtained by minimizing the discrepancy between the observed values in the system and the forecasted or modelled state of the system within a certain time interval. In the eighties and the nineties, several articles concerning inverse analysis in geomechanics using the Maximum Likelihood and the (Extended) Bayesian method were published (Gens et al. 1996, Honjo et al. 1994a,b). Recent developments in other fields of science have shown a new powerful technique indicated as the Ensemble Kalman filter. In a filter, the state of the system is analysed each time data becomes available.

1.2 *Geological uncertainty*

Usually in the field of geomechanics the Finite Element Method is used for numerical simulations. In this method, one particular value for a certain parameter is assigned to the soil mass or a soil layer, which remains constant throughout the mass or the layer. However, in nature, this is not really the case: properties of natural soils will vary through depth and often also in horizontal extent due to for example different loading conditions or different depositional conditions. One way to deal with this uncertainty is the Random Finite Element Method. This technique incorporates the spatial correlation between the properties using Monte Carlo simulations in order to represent the proper stochastic properties.

The general formulations of the Ensemble Kalman filter and the Random Finite Element Method will be discussed in the next sections. In Hommels et al. 2005 the effectiveness of the Ensemble Kalman filter using the Finite Element Method has been proven. In this paper the Ensemble Kalman filter will be combined with the Random Finite Element method in a conceptual case study. The sensitivity of the analyses to the number of Monte Carlo simulations is also considered

2 THEORY

2.1 *Basic principles*

The true state of the subsurface at time step k can be described by a state vector $x^t(k)$. The elements of the state vector are filled with stresses but possibly also strains or other state parameters. The superscript 't' denotes that $x^t(k)$ is the "true" state; the exact value is probably unknown. To obtain insight in the true state, a model is developed to make a forecast or to model $x^f(k+1)$ at time step $k+1$:

$$x^f(k+1) = M(x^f(k)) \tag{1}$$

The superscript 'f' denotes that $x^f(k)$ is a "forecast" of the true state $x^t(k)$ at time step k, which is in the best case a good approximation. In the context of the shallow subsurface, the state vector can partly be filled with displacements u. M denotes the dynamical model operator, which describes for instance the constitutive model of the soil, e.g. the soil parameters Young's modulus E and Poisson's ratio ν in the simplest elastic case. If there are uncertainties in the parameters, which have to be updated, the state vector $x^f(k)$ is also filled with the uncertain parameters.

Since models are never perfect:

$$x^t(k) = x^f(k) + \eta(k) \tag{2}$$

in which $\eta(k)$ is the unknown model error in the k-th forecast with $E\{\eta\}=0$ (E denotes expectation) and $E\{\eta^2\}= P$, which is the model error covariance matrix.

Some entities of the state are compared with data from an observational network, for example the measurements y^o of the surface displacements. All available data for time step k are stored in an observation vector $y^o(k)$. The superscript 'o' denotes that $y^o(k)$ is an "observation". There is a difference $\varepsilon(k)$ between the "true" state $x^t(k)$ and the actual "observed" data $y^o(k)$:

$$y^o(k) = H(k)x^t(k) + \varepsilon(k) \tag{3}$$

where $H(k)$ is the linear observational operator. The observations are assumed unbiased $(E\{\varepsilon\}=0))$ and $E(\varepsilon^2)= R$, which is the measurement error covariance matrix.

The final goal of inverse modelling methods is to improve the state vector; at each time step k measurements become available, with an error $\varepsilon(k)$ as small as possible.

2.2 *Extended Kalman filter*

In a filter, the state of the system is analysed each time data becomes available. This is done in a two-step approach. In the first step, the time update step, the modelled values are calculated for the time measurements will come in. Also the forecasted (or modelled) error covariance matrix P^f is calculated as well as the Kalman gain K, which is a kind of weighting factor between the measured values and the modelled values. The Kalman gain K is independent of the measurements. In the second step, the measurement update step, the measurements will come in and the assimilated state vector $x^a(k)$ using the Kalman gain are calculated as well as the assimilated error covariance matrix P^a. The simplest filter is the linear Kalman filter. However the linear Kalman filter is designed for linear systems and therefore not very useful in geomechanics. It is possible to derive approximate filters using linearization techniques for non-linear systems. Again, assume that the system can be described by a state vector x, and the physical process is governed by the non-linear stochastic equation:

$$x^f(k+1) = f(x^t(k),u(k)) + G(k)w(k) \tag{4}$$

in which $u(k)$ is the system input, $G(k)$ is the noise input matrix and $w(k)$ is the process noise with zero mean and covariance matrix Q. In the time update step, the equations can be written as:

$$x^f(k+1) = f(x^a(k),u(k)) \tag{5}$$

$$P^f(k+1) = F(k)P^a(k)F^T(k) + G(k)Q(k)G^T(k) \tag{6}$$

where

$$F(k) = \left(\frac{\partial f}{\partial x}\right)_{x_{ref}} \tag{7}$$

The measurement update equations can be written as follows:

$$\begin{aligned} x^a(k+1) &= x^f(k+1) \\ &\quad + K(k+1)\left\{y^o(k+1) - H(k+1)x^f(k+1)\right\} \end{aligned} \tag{8}$$

$$P^a(k+1) = \left\{I - K(k+1)H(k+1)\right\} P^f(k+1) \tag{9}$$

And the Kalman gain is given by:

$$K(k+1) = \frac{P^f(k+1)H^T(k+1)}{H(k+1)P^f(k+1)H^T(k+1) + R(k+1)} \tag{10}$$

2.3 *Ensemble Kalman Filter*

Evensen introduced the Ensemble Kalman filter (EnKF) in 1994 and the theoretical formulations as well as an overview of several applications are described in Evensen (1994, 2003). The EnKF was designed to resolve two major problems related to the use of the extended Kalman filter (EKF). The first problem relates to the use of an approximate closure scheme in the EKF, and the other one to the huge computational requirements associated with the storage and forward integration of the error covariance matrix *P*. For further details the reader is referred to the references.

In the Ensemble Kalman filter, an ensemble of possible state vectors, which are randomly generated using a Monte Carlo approach, represents the statistical properties of the state vector. The algorithm does not require a tangent linear model, which is required for the EKF, and is very easy to implement.

At initialisation, an ensemble of N initial states $(\xi_N)_0$ are generated to represent the uncertainty at time step $k=0$. In the measurement update step, the matrix $E^f(k+1)$ defines an approximation of the covariance matrix $P(k+1)$.

The time update equations for the Ensemble Kalman filter for each estimate of the state are:

$$\xi_i^f(k+1) = f(\xi_i^a(k)) + G(k)w(k) \tag{4}$$

$$x^f(k+1) = \frac{1}{N}\sum_1^N \xi_i^f(k+1) \tag{5}$$

$$\begin{aligned} E^f(k+1) = [&\xi_1^f(k+1) - x^f(k+1), \ldots, \\ &\xi_N^f(k+1) - x^f(k+1)] \end{aligned} \tag{6}$$

In the measurement update step:

$$P(k+1) = \frac{1}{N-1} E^f(k+1)(E^f(k+1))^T \tag{7}$$

Equation (4) is never really calculated due to huge computational requirements, as mentioned before, but is split up:

$$K(k+1) = \frac{E^f((E^f)^T H)}{(N-1)\left(H\frac{1}{N-1}E^f((E^f)^T H^T) + R\right)} \tag{8}$$

$$\begin{aligned} \xi_i^a(k+1) &= \xi_i^f(k+1) \\ &+ K\left\{y_o(k+1) - H\xi_i^f(k+1) + e\right\} \end{aligned} \tag{9}$$

where ε is for each ensemble member a randomly added measurement error.

2.4 *Random Finite Element Method*

The Random Finite Element Method (RFEM) combines the finite element analysis with the random field theory generated using the local average subdivision method (Fenton & Vanmarcke 1990). A brief description will be given.

In the Finite Element Method the uncertainty of a material is defined by its mean μ and its standard deviation σ. For the introduction of more spatial variability, the introduction of an additional statistical parameter, the spatial correlation length θ, is required. The spatial correlation length defines the distance beyond which there is minimal correlation and can be determined from for example CPT-data. A large value of θ indicates a strongly correlated material, while a small value indicates a weakly correlated material. Now the random field can be generated and in the Random Finite Element Method, this is done using the local average subdivision (LAS) (Fenton & Vanmarcke 1990), based on a standard normal distribution (mean μ is zero and the standard deviation σ equals one) and a spatial correlation function ρ. LAS generates a square random field by uniformly subdividing a square domain into smaller square cells, where each cell has a unique local average, which is correlated with surrounding cells. For an isotropic field, where θ is equal in all directions, the Gauss-Markov correlation function ρ is given by

$$\rho(\tau) = \exp\left(-\frac{2}{\theta}|\tau|\right) \tag{10}$$

in which τ is the lag vector. The random field for a certain parameter E for cell i, based on a standard normal distribution, can then be transformed into for example a normal distribution:

$$E_i = \mu_E + \sigma_E Z_i \tag{11}$$

in which μ_E is the mean and σ_E is standard deviation of parameter E; Z_i is the local average value for cell i.

There are an infinite number of possibilities for the random field, based on the given set of statistics of μ, σ and θ. Again Monte Carlo simulations are used to express the spatial distribution (Hicks & Samy 2002).

2.5 *Monte Carlo simulations*

Since both the Ensemble Kalman filter and the Random Finite Element Method are based on Monte Carlo simulation, these simulations are combined in order to save computational effort. However enough realizations are required to ensure a good representation of probability density of the state estimate.

3 CASESTUDY

3.1 *Loaded soil column*

Consider a soil column consisting of 16 elements, which are 1 meter in length. Each element has a random stiffness EA, calculated using LAS. The column is axially loaded at element number 16.

The true state value of the element axial stiffness vector *EA* is 16 KN and based on this value the measurements are generated. The measurement noise is normally distributed with zero mean and standard deviation 0.01.

The stiffness *EA* of the soil has a normal distribution, with mean $\mu = 16$ *KN* and standard deviation $\sigma = 2$. The total length of the column is 16 meter and the spatial correlation length θ equals 3. During 100 timesteps the load is increased from 0.5 kN to 1.1 kN.

3.2 *Update process*

In figure 1, the results of the update of the stiffness, for one element is shown. The assimilated values tend to approach the value of the true state, but it takes a while due to the random field assumption.

At the start of the update or assimilation process, the uncertainty of parameter EA is very large. The range of the ensemble members is shown in figure 2. In figure 3, the range of the ensemble members is shown after 100 loading steps. One can see that there has been a large reduction of the parameter uncertainty. The true state can be found, for both loading steps, between the range of the ensemble members and therefore the filter is performing well. In figure 4, the different distributions for loading step 1 and loading step 100 are shown. Again the large reduction of the uncertainty is shown. The distribution at loading step 100 is much smaller than the distribution at time step 1. Therefore the uncertainty about the value of the stiffness *EA* has decreased.

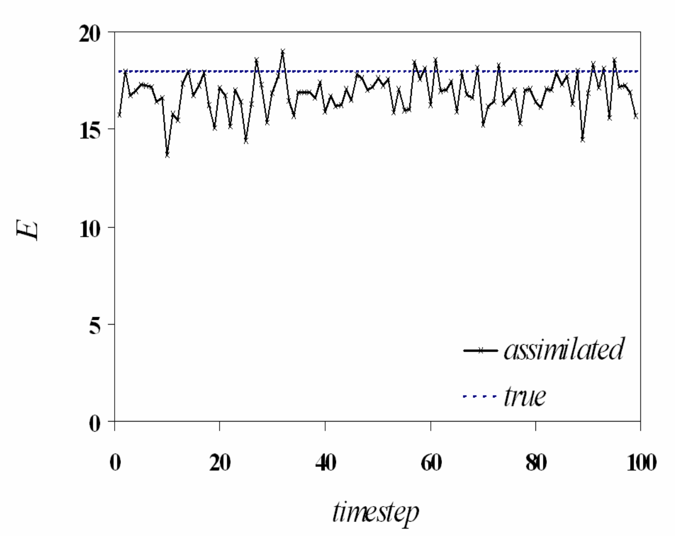

Figure 1: the update process of parameter *E* for several time steps

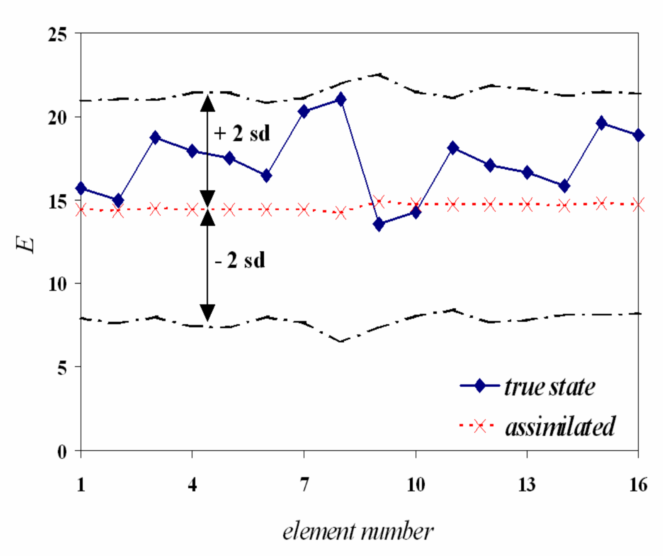

Figure 2: uncertainty of parameter *E* at loading step 1

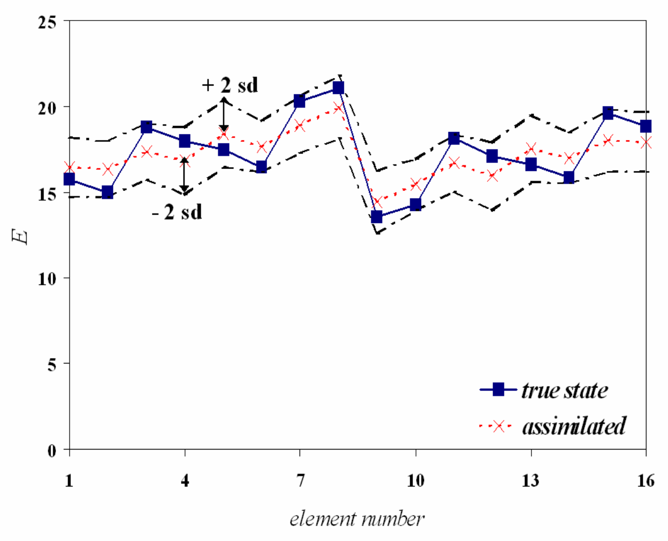

Figure 3: uncertainty of parameter *E* after 100 loading steps

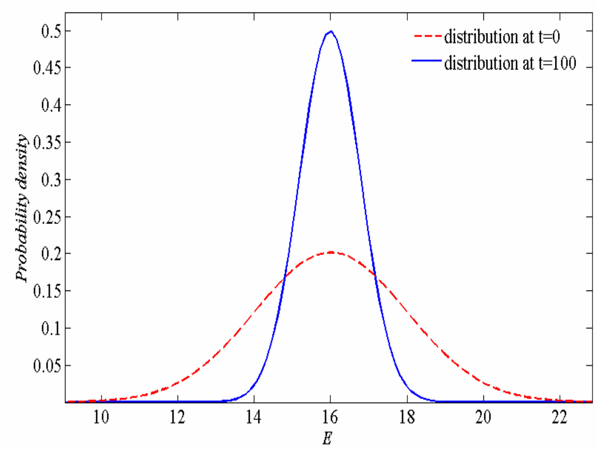

Figure 4: comparison of starting and final distribution of E

3.3 *Sensitivity to the number of MC realizations*

In principle, a higher number of Monte Carlo realisations will give a better estimate of the parameter. However often not a very large amount is required to obtain good results. In figure 5 the update of each element after 100 loading steps for a different number of Monte Carlo realizations is shown. A number of 500 MC realizations gives a slightly better estimate than for 100 MC realizations. However, the difference is very minimal. One can then choose for a lower amount of MC realizations, which will save computation time.

In figure 5 it is also shown that the true state is never fully reached, not even for 500 MC realizations. This is due the fact that the observation points are limited to 4 points. If the observation points were "traveling" throughout the material, the estimation would be much better, because then the true state is observed at least once. However this is hardly ever the case in the "real" world and therefore not simulated here.

3.4 *Embankment*

The authors are working on an a real case study of the construction of a road embankment where the Ensemble Kalman filter in combination with the Random Finite Element Method will be applied to determine several uncertain parameters of the shallow subsurface. During the presentation the first results will be shown.

4 CONCLUSIONS

The Ensemble Kalman filter has already proven to be effective in the field of geomechanics using the standard finite element code (Hommels et al. 2005). In this paper the effectiveness of the ensemble Kalman filter in combination with the Random Finite Element Method is shown.

After 100 loading steps, the parameter uncertainty has been decreased significantly. The true state will never be fully reached if there are a limited amount of observation points at fixed places.

The analysis is slightly sensitive to the amount of Monte Carlo realizations. It is shown that not a very high amount of realizations is necessary to obtain a good representation of the probability density of the state estimate.

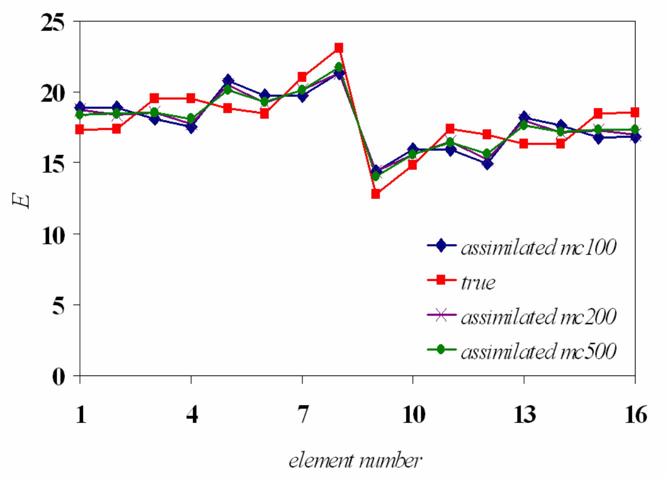

Figure 5: comparison of different amount of Monte Carlo simulations

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