Comparison of the Ensemble Kalman filter with the Unscented Kalman filter: application to the construction of a road embankment

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ABSTRACT

The Extended Kalman Filter (EKF) has been used in the field of geomechanics for nonlinear state space estimation (Murakami, 1991). However a couple of alternative approaches have emerged over the last few years, namely the Ensemble Kalman filter (EnKF) and the Unscented Kalman filter (UKF).

The EnKF was designed to resolve two major problems related to the use of EKF. The first problem relates to the use of an approximate closure scheme in the EKF (first order Taylor expansion). The second problem relates to the huge computational requirements associated with the storage and forward integration of the error covariance matrix *P*. In the Ensemble Kalman filter, an ensemble of possible state vectors, which are randomly generated using a Monte Carlo approach, represents the statistical properties of the state vector. The EnKF algorithm does not require a tangent linear model, which is required for the EKF, and is very easy to implement.

The UKF claims a higher accuracy and robustness for non-linear models than the EKF. Instead of linearizing the functions as is done in the EKF, the UKF uses a set of points and propagates this set through the actual non-linear function. These points are chosen such that their mean, covariance and possibly also higher order moments match the Gaussian random variable. The mean and the covariance can be recalculated from the propagated points, yielding more accurate results compared to the ordinary function linearization.

The performance of the EnKF compared to the UKF will be shown in a conceptual nonlinear case study, based on the construction of a road embankment.

Keywords:inverse modelling, Kalman filtering, parameter estimation

INTRODUCTION

In 1960, Kalman published a paper in which he describes a recursive solution to the discrete data linear filtering problem (Kalman, 1960). In a filter, the state of the system is analyzed each time data becomes available.

Although the linear Kalman filter was designed for linear systems, it is possible to derive approximate filters using a linearization techniques for non-linear systems. The Extended Kalman Filter is such a filter and has been used in the field of geomechanics for nonlinear state space estimation (Murakami, 1991). Over the last few years, a couple of alternative approaches have emerged, namely the Ensemble Kalman filter (EnKF) and the Unscented Kalman filter (UKF).

Evensen introduced the EnKF in 1994 and the theoretical formulations as well as an overview of several applications are described in Evensen (2003). The EnKF was designed to resolve two major problems related to the use of EKF. The first problem relates to the use of an approximate closure scheme in the EKF (first order Taylor expansion). The second problem relates to the huge computational requirements associated with the storage and forward integration of the error covariance matrix *P*. For further details the reader is referred to the references. In the Ensemble Kalman filter, an ensemble of possible state vectors, which are randomly generated using a Monte Carlo approach, represents the statistical properties of the state vector. The algorithm does not require a tangent linear model, which is required for the EKF, and is very easy to implement.

The UKF was first proposed by Julier and Uhlman (1997) and further developed by Wan and Van der Merwe (2001). Instead of linearizing the functions as is done in the EKF, the unscented transformation uses a set of points, and propagates these points through the actual nonlinear function. The points are chosen such that their mean, covariance and possibly also higher order moments match the Gaussian random variable. The mean and the covariance can be recalculated from the propagated points, yielding more accurate results compared to the ordinary function linearization.

Both the EnKF and the UKF have their advantages and disadvantages. The good performance of the EnKF has been shown in Hommels et al. (2005). The UKF has not been introduced in the field of geomechanics.

In the following sections, the EKF, EnKF and the UKF are introduced. In a conceptual nonlinear casestudy, based on the construction of a road embankment, the performances of the EnKF and the UKF are compared to each other. The measurements of the vertical displacement at several observation points were used to improve the uncertainty in the model state and theYoung's modulus *E*.

KALMAN FILTERING

Extended Kalman Filter

The linear Kalman filter is designed for linear systems. It is, however, possible to derive approximate filters using linearization techniques for non-linear systems. Assume that the system can be described by a state vector *x*, the physical process is then governed by the nonlinear stochastic equation:

$$x^f(k+1) = f(x^t(k), u(k)) + G(k)w(k) \quad (1)$$

in which x^f is the forecasted state vector, $u(k)$ is the system input at time k, $G(k)$ is the noise input matrix and $w(k)$ is the process noise with covarince matrix Q (equation 3).

In the time update step, the equations can now be written as:

$$x^f(k+1) = f(x^a(k), u(k)) \quad (2)$$

in which $x^a(k)$ is the assimilated state vector.

The forecasted error covariancematrix P^f is calculated as follows:

$$P^f(k+1) = F(k)P^a(k)F^T(k) + G(k)Q(k)G^T(k) \tag{3}$$

in which

$$F(k) = \left(\frac{\partial f}{\partial x}\right)_{x_{ref}} \tag{4}$$

and P^a is the assimilated error covariance matrix. The measurement update equations can be written as follows:

$$x^a(k+1) = x^f(k+1) + K(k+1)\left\{y^o(k+1) - H(k+1)x^f(k+1)\right\} \tag{5}$$

in which y^o is the vector, which contains the observations.

$$P^a(k+1) = \left\{I - K(k+1)H(k+1)\right\}P^f(k+1) \tag{6}$$

and the Kalman gain

$$K(k+1) = \frac{P^f(k+1)H^T(k+1)}{H(k+1)P^f(k+1)H^T(k+1) + R(k+1)} \tag{7}$$

in which H is the observational operator

Unscented Kalman filter

In 1997, Julier and Uhlmann introduced the Unscented Kalman filter (UKF) using the unscented transformation. Later, it was analyzed more in depth by Wan and Van der Merwe (2001). The UKF is based on the principle that is easier to approximate a Gaussian distribution than it is to approximate arbitrary nonlinear functions. They have shown that the UKF leads to more accurate results than the EKF and that the UKF generates much better estimates of the covariance of the states (the EKF seems to underestimate this quantity).

The UKF adresses the major shortcomings of the EKF by drawing a set of sigma-points from the distribution of the state vector $\boldsymbol{x}$ at the beginning of the time step and then propagating them in time via the nonlinear mapping. If these sigma-points are chosen appropriately, it can be shown that the mean and covariance of $\boldsymbol{x}$ at the end of the time step are at the third order for a Gaussian distribution. The number of sigma points equals $2L+1$, where L is the dimension of the state variable x of the state vector $\boldsymbol{x}$. If x is a scalar, the number of sigma points equals 5.

Selection of the sigma points

A matrix $\boldsymbol{\chi}$ of $2L+1$ sigma vectors χ_i is calculated as follows:

$$\begin{cases} \chi_0 = \bar{x} \\ \chi_i = \bar{x} + \left(\sqrt{(L+\lambda)P_x}\right)_i & i = 1,\ldots,L \\ \chi_i = \bar{x} - \left(\sqrt{(L+\lambda)P_x}\right)_{i-L} & i = L+1,\ldots,2L \end{cases} \tag{8}$$

in which $\lambda = \alpha^2(L+\kappa) - L$ is a scaling parameter. The constant α determines the spread of the sigma points around $\bar{x}$ and is usually set to a small positive value (e.g., $1 \le \alpha \le 1e-4$). The constant κ is a secondary scaling parameter which is usually set to 0 or 3 – L. The constant β is used to incorporate prior knowledge of the distribution $\mathbf{x}$ (for Gaussian distributions, $\beta = 2$ is optimal).

Algorithm

At initialisation:

$$\bar{x}_0 = E[x_0] \tag{9}$$

$$P_0 = E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T] \tag{10}$$

$$\bar{x}_0^{aug} = E[x^{aug}] = \begin{bmatrix} \bar{x}_0^T & 0 & 0 \end{bmatrix} \tag{11}$$

$$P_0^{aug} = E[(x_0^{aug} - \bar{x}_0^{aug})(x_0^{aug} - \bar{x}_0^{aug})^T] = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \tag{12}$$

In equations 11 and 12 superscript 'aug' indicates augmented state vector $\bar{x}_0^{aug}$ and augmented error covariance matrix P_0^{aug}. Q and R are the process noise covariance matrix and the measurement covariance matrix respectively.

Calculate the augemented sigma points χ_{k-1}^{aug}:

$$\chi_{k-1}^{aug} = \begin{bmatrix} \bar{x}_{k-1}^{aug} & \bar{x}_{k-1}^{aug} \pm \sqrt{(L+\lambda) P_{k-1}^{aug}} \end{bmatrix} \tag{13}$$

The time update equations are as:

$$\chi_k^f = f(\chi_{k-1}^a) \tag{14}$$

$$\bar{x}_k^f = \sum_{i=0}^{2L} W_i^m \chi_{i,k}^f \tag{15}$$

$$P_k^f = \sum_{i=0}^{2L} W_i^c \left\{ \chi_{i,k}^f - \bar{x}_{i,k}^f \right\} \left\{ \chi_{i,k}^f - \bar{x}_k^f \right\}^T \tag{16}$$

$$\Upsilon_k^f = h(\chi_k^f) \tag{17}$$

$$\bar{y}_k^f = \sum_{i=0}^{2L} W_i^m \Upsilon_{i,k}^f \tag{18}$$

in which $\bar{x}_k^f$ is the forecasted mean of the sigma points. W_i^m and W_i^c are the weights as calculated in equation 24.

The measurement update equations are as follows:

$$P_{yy}^a = \sum_{i=0}^{2L} W_i^c \left\{ \Upsilon_{i,k}^f - \bar{y}_k^f \right\} \left\{ \Upsilon_i^f - \bar{y}_k^f \right\}^T \tag{19}$$

$$P_{xy}^a = \sum_{i=0}^{2L} W_i^{(c)} \left\{ \chi_{i,k}^f - \bar{x}_k^f \right\} \left\{ \Upsilon_{i,k}^f - \bar{y}_k^f \right\}^T \tag{20}$$

$$\mathrm{K}_k = P_{yy}^a P_{xy}^a \tag{21}$$

$$x_k^a = x_k^f - K_k (y_k^o - \bar{y}_k^f) \tag{22}$$

$$P^a = P^f - K_k P_{yy} K_k^T \tag{23}$$

The weights W_i are determined by

$$\begin{cases} W_0^{(m)} = \dfrac{\lambda}{L+\lambda} \\ W_0^{(c)} = \dfrac{\lambda}{L+\lambda} + (1-\alpha^2+\beta) \\ W_i^{(m)} = W_i^{(m)} = \dfrac{1}{2(L+\lambda)} \quad i = 1,\ldots,2L \end{cases} \tag{24}$$

Ensemble Kalman filter

Evensen introduced the Ensemble Kalman filter in 1994 and the theoretical formulations as well as an overview of several applications are described in Evensen (1994 and 2003). The EnKF was designed to resolve two major problems related to the use of the extended Kalman filter, which are mentioned earlier.

In the Ensemble Kalman filter, an ensemble of possible state vectors, which are randomly generated using a Monte Carlo approach, represents the statistical properties of the state vector. The algorithm does not require a tangent linear model, which is required for the EKF, and is very easy to implement.

At initialisation, an ensemble of N initial states $(\xi_N)_0$ are generated to represent the uncertainty at time step $k=0$. In the measurement update step, the matrix $E^f(k+1)$ defines an approximation of the assimilated covariance matrix $P^a(k+1)$.

The time update equations for the EnKF for each estimate of the state are:

$$\xi_i^f(k+1) = f(\xi_i^a(k)) + G(k)w(k) \tag{25}$$

$$x^f(k+1) = \frac{1}{N}\sum_{1}^{N}\xi_i^f(k+1) \tag{26}$$

$$E^f(k+1) = [\xi_1^f(k+1) - x^f(k+1),\ldots,\xi_N^f(k+1) - x^f(k+1)] \tag{27}$$

In equations 25 and 26, $G(k)$ and $w(k)$ are the noise input matrix and process noise vector respectively; x^f is the 'forecasted' state vector.

In the measurement update step, the assimilated error covariance matrix $P^a(k+1)$ can be calculated as follows:

$$P^a(k+1) = \frac{1}{N-1}E^f(k+1)(E^f(k+1))^T \tag{28}$$

Equation 28 is never really calculated due to huge computational requirements, but is split up:

$$K(k+1) = \frac{E^f((E^f)^T H)}{N-1}\left[\frac{HE^f}{N-1}((E^f)^T H^T) + R\right]^{-1} \tag{29}$$

$$\xi_i^a(k+1) = \xi_i^f(k+1) + K\left\{y_o(k+1) - H\xi_i^f(k+1) + \varepsilon\right\} \tag{30}$$

In equation 29, K is the Kalman gain, H is the observational operator and R is the measurement noise covariance matrix with zero mean. In equation 30: y_o is the vector, which contains the observations at time step $k+1$ and ε is an additional noise (Evensen, 2003).

Overview

In figure 1 a schematic diagram of the three different methods is shown. On the left, the EnKF has an ensemble of state vectors propagated through the non linear function $f(x)$, and the true posterior mean and variance are calculated. In the middle, the EKF, the random variables are calculated by a linearization approach. On the right, the UKF is shown, where 5 sigma points are propagated through the non linear function $f(x)$.

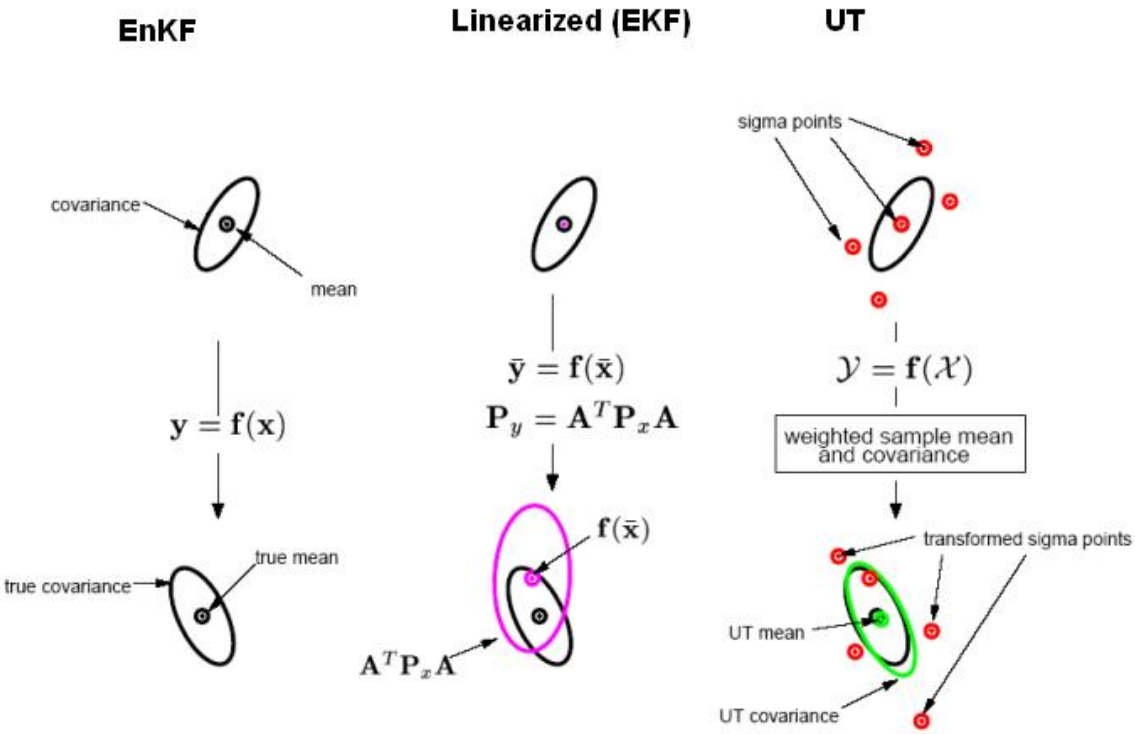

Figure 1. Comparison of the EnKF (Monte Carlo), the EKF and the Unscented Transformation, which serves as a basis for the UKF (van der Merwe et al., 2000)

CASE STUDY

Construction of a road embankment

Consider the construction of a four meter high embankment on soft clay (left of figure 2). A dedicated model for the determination of the settlement has been written using Smith and Griffiths (2004). In the left of figure 2 the three-layered foundation below the embankment is shown. The nodes at which the observations of the vertical displacement were performed, are indicated with a triangle. The properties of the foundation, including the uncertain Young's modulii E are given in table 1. For the material model the purely elasto-plastic Mohr-Coulomb model is used. The groundwater table coincides with the original ground surface.

The embankment is built in four lifts and at the end of each lift a consolidation period is considered. The graph in the right of figure 2 shows at the left axis the different stages during the construction versus the total time, while at the right axis the measured displacement versus the total time at the end of each lift at a certain observation point, is shown. The measurements were generated based on an assumed true state.

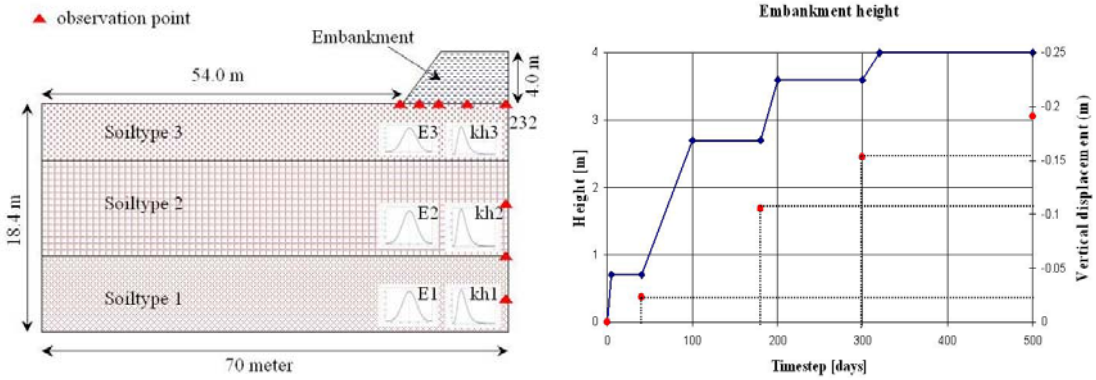

Figure 2. (left) Three-layered foundation; the observation points are indicated with a red triangle; (right) embankment height (left axis) and vertical displacement (right) during the construction phases

Table 1. Properties of the foundation below the embankment

Soil layer	μ_E (kPa)	σ_E	ν	c	φ	ψ	γ(kN/m^3)	k_h(m/day)	k_v(m/day)
1	*3400*	*340*	*0.33*	*16*	*12.5*	*0*	*16.5*	*2.41E-05*	*2.94E-05*
2	*1400*	*140*	*0.33*	*22*	*13.5*	*0*	*15.6*	*8.34E-05*	*8.99E-05*
3	*4600*	*460*	*0.30*	*8*	*6.5*	*0*	*17.3*	*3.56E-04*	*1.03E-04*

RESULTS AND CONCLUSIONS

Figures 3,4 and 5 show the results of the update process of the Young's modulii for soil layers 1,2 and 3 respectively using the EnKF and the UKF. The dotted horizontal line is the true value of the Young's modulus *E* in each layer. The results of the updated process using the EnKF are indicated with a '+'; the results of the updated process using the UKF are indicated with a 'x'.

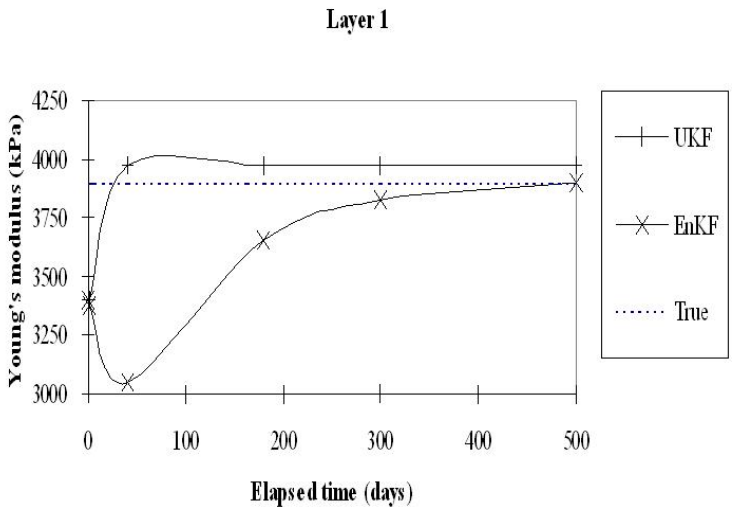

Figure 3. Update of Young's modulus *E* of layer 1

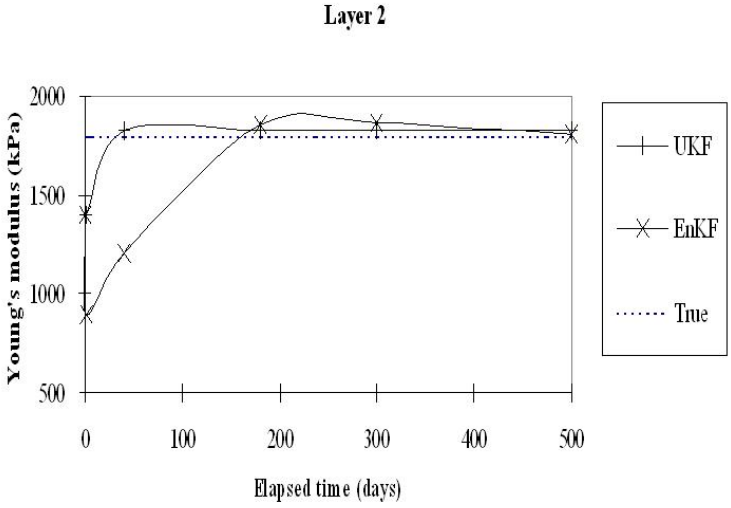

Figure 4. Update of Young's modulus *E* of layer 2

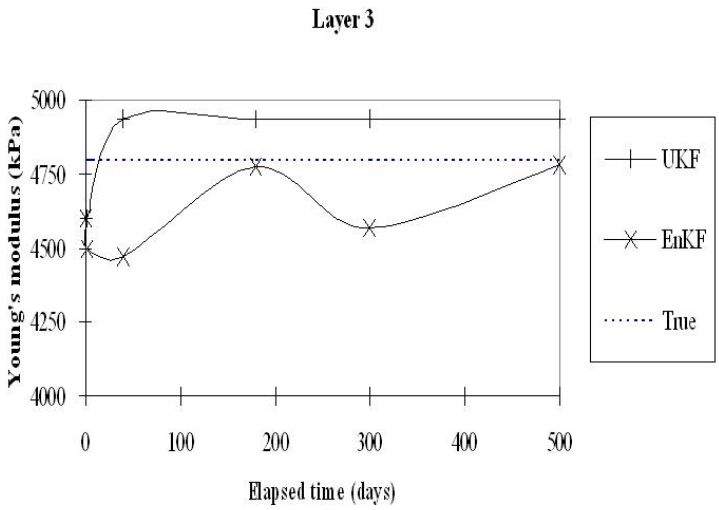

Figure 5. Update of Young's modulus E of layer 3

From figures 3 and 5 it is clear that the Ensemble Kalman filter performs much better than the Unscented Kalman filter. The UKF overestimates the Young's modulus in layers 1 and 3 and doesn't change much after one update, which probably implies that the filter is quite sure about its estimation. This can be seen in figure 6, which shows the standard deviation for each soil layer for the UKF. In figure 7, the standard deviation of the update process of the Young's moduli E of the three soil layers using the EnKF are shown. From figure 7 it is clear that at the start of the update process the filter is not quite sure about his estimation, but at the end of the consolidation period, when the true state is reached, as can be seen in figures 3 and 5, the standard deviation reaches zero.

Figure 4 shows more or less the same performance using the EnKF and the UKF; at the final update step the EnKF has a better value than the UKF, which again is sure about his estimation (figure 6). This behaviour can be explained by the presence of only one observation point in layer 2, as can be seen in the left of figure 2.

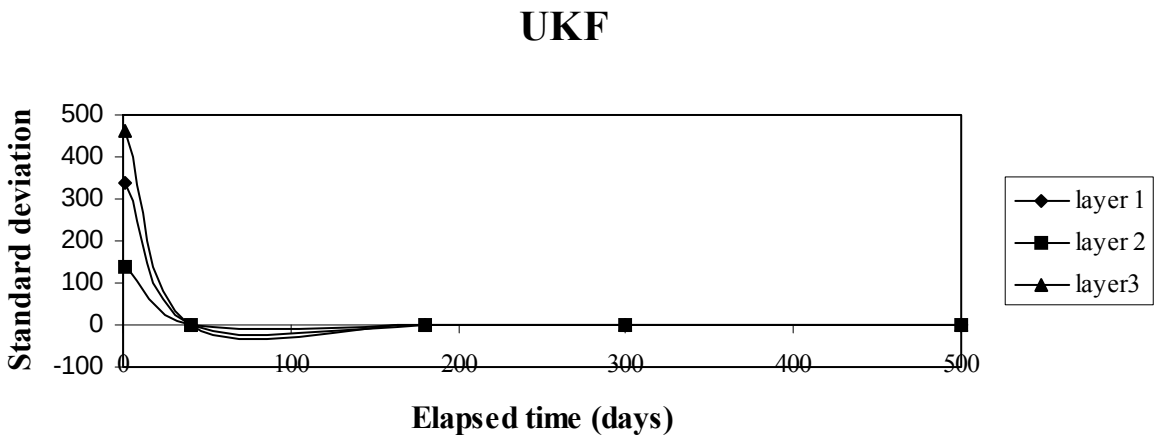

Figure 6. Standard deviation of the Young's moduli during time of the three soil layers using the UKF

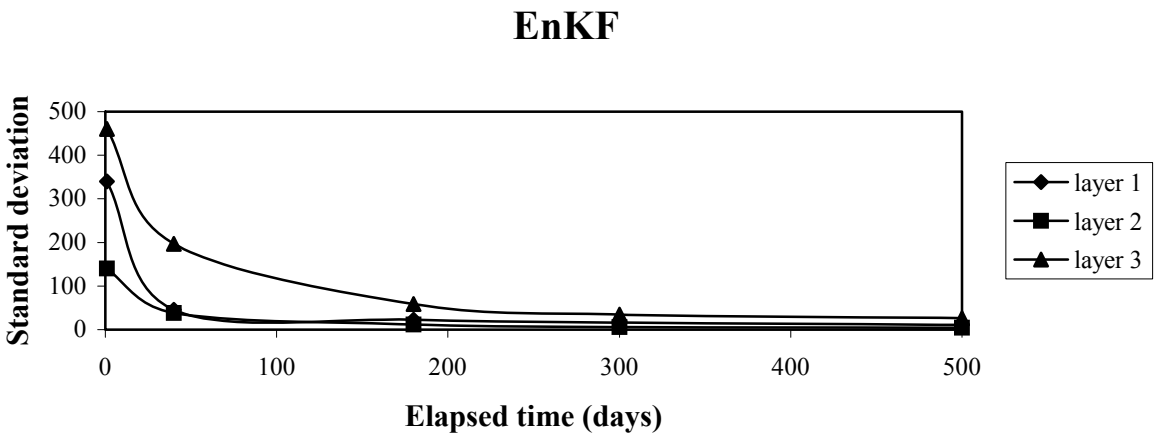

Figure 7. Standard deviation of the Young's moduli during time of the three soil layers using the EnKF

The computational time for both filters was the same.

From the above results it can be concluded that the EnKF performs much better than the UKF. For soil layers 1 and 3 this is more evident than for soil layer 2.

ACKNOWLEDGEMENTS

This research has been performed on the basis of the invitatio program for short period, provided by the Japanese Society for Promotion of Science (JSPS).
The authors are very grateful to the efforts of Satoshi Kasamutsu, who performed his MSc-thesis on this subject.

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